Solving Systems of Linear Equations

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Example:

3 resources R_1, R_2, R_3

4 different products P_1, P_2, P_3, P_4

$$\boldsymbol{A} = \begin{array}{c|cccc} & P_1 & P_2 & P_3 & P_4 \\ \hline R_1 & 1 & 1 & 2 & 1 \\ R_2 & 2 & 1 & 1 & 0 \\ R_3 & 1 & 0 & 1 & 2 \end{array} \qquad \boldsymbol{b} = \begin{array}{c|c} R_1 & 8 \\ R_2 & 6 \\ R_3 & 8 \end{array}$$

Question: How many products can be produced if we have b many resources in stock?

Example:

3 resources R_1, R_2, R_3

4 different products P_1, P_2, P_3, P_4

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 8 \end{pmatrix}$$

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$$

Question: How many products can be produced if we have b many resources in stock?

Gaussian elimination algorithm
Objective: Solve $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$

1. Step Create augmented matrix $\left[\boldsymbol{A}\,\middle|\,\boldsymbol{b}\right]$
2. Step Use one of the following operations
   (a) Swap positions of two rows
   (b) Multiply one row by a non-zero scalar factor
   (c) Add one row to a (scalar multiple) of another row

   to obtain (reduced) row echelon form
3. Step Obtain the solutions of the equation

Reduced row echelon form

$$\left(\begin{array}{ccccccc|c}
1 & * & 0 & * & \cdots & 0 & * & b_1 \\
0 & 0 & 1 & * & \cdots & 0 & * & b_2 \\
0 & 0 & 0 & 0 & \cdots & 0 & * & b_3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & * & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & * & b_{r-1} \\
0 & 0 & 0 & 0 & \cdots & 1 & * & b_r \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{r+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_n
\end{array}\right)$$

Reduced row echelon form

$$\left(\begin{array}{ccccccc|c} \boxed{1} & * & 0 & * & \cdots & 0 & * & b_1 \\ 0 & 0 & \boxed{1} & * & \cdots & 0 & * & b_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & b_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & * & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & b_{r-1} \\ 0 & 0 & 0 & 0 & \cdots & \boxed{1} & * & b_r \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_n \end{array}\right)$$

- r **pivots** ($\boxed{1}$)

Reduced row echelon form

$$
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- r **pivots** ($\boxed{1}$)
- All elements *below, above and left* of pivots are zeros
- The number of pivots $0 \leq r \leq n$ is called **rank** of the matrix A

Question: How many solutions does the System $Ax = b$ have?

Reduced row echelon form:

$$
n \text{ rows} \left\{ \underbrace{\left(\begin{array}{ccccccc|c}
\boxed{1} & & 0 & & \cdots & 0 & & b_1 \\
0 & 0 & \boxed{1} & & \cdots & 0 & & b_2 \\
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- No solution if $r < n$ and $b_i \neq 0$ for $i \in \{r+1, \ldots, b_n\}$

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- Unique solution if $r = m$ and $b_{r+1} = \ldots = b_n = 0$
- Infinitely many solutions if $r < m$ and $b_{r+1} = \ldots = b_n = 0$

Examples:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right) \longrightarrow \begin{array}{ll} x_1 & = 2 \\ x_2 & = 3 \\ x_3 & = 4 \end{array} \longrightarrow \text{Unique Solution}$$

3 pivot variables / rank 3

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 42 \end{array}\right) \longrightarrow 0 = 42 \longrightarrow \text{Contradiction}$$

2 pivot variables / rank 2

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{ll} x_1 & = 1 + x_3 \\ x_2 & = 2 - x_3 \end{array} \longrightarrow \infty\text{-many solutions}$$

2 pivot variables / rank 2

Consider the following reduced row echelon form

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Solution Space:

$$L = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

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Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a quadratic matrix.
A matrix $\boldsymbol{A}^{-1}$ is called **Inverse Matrix** of $\boldsymbol{A}$ if

$$\boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}_n \qquad \text{and} \qquad \boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{I}_n$$

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Then

$$\begin{aligned} && \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \Leftrightarrow && \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{A}^{-1}\boldsymbol{b} \\ \Leftrightarrow && \boldsymbol{I}_n\boldsymbol{x} &= \boldsymbol{A}^{-1}\boldsymbol{b} \\ \Leftrightarrow && \boldsymbol{x} &= \boldsymbol{A}^{-1}\boldsymbol{b} \end{aligned}$$

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Theorem

$\boldsymbol{A}^{-1}$ *exists* $\Leftrightarrow$ $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ *has a unique solution* $\Leftrightarrow$ *The rank of* $\boldsymbol{A}$ *is* n.

Computation of the Inverse

- Create the augmented Matrix $\left[\boldsymbol{A} \,\middle|\, \boldsymbol{I}_n\right]$
- Compute the reduced row echelon form
- If the reduced form is $\left[\boldsymbol{I}_n \,\middle|\, \boldsymbol{B}\right]$ then $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$

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Example:

$$\boldsymbol{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

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$$\left[\boldsymbol{A} \,\middle|\, \boldsymbol{I}_n\right] = \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

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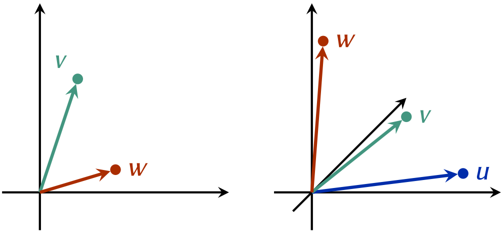
$$\boldsymbol{A}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

A family of k vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ is called **linear independent** if the only solution of

$$\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 + \ldots + \boldsymbol{v}_k x_k = 0$$

is $x_1 = x_2 = \ldots = x_k = 0$.



linear independent vectors

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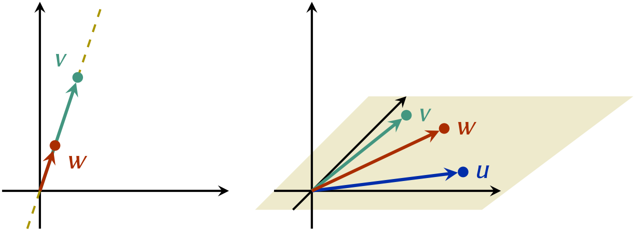

linear dependent vectors

v, w lie on a line

u, v, w lie on plane

A family of k vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ is called **linear independent** if the only solution of

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is $x_1 = x_2 = \ldots = x_k = 0$.
The vectors are linear independent if the matrix

$$\begin{pmatrix} v_{1,1} & v_{2,1} & \cdots & v_{k,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{k,2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1,n} & v_{2,n} & \cdots & v_{k,n} \end{pmatrix}$$

has rank k.